Exact solution of equations governing aligned plane rotating magnetohydrodynamics steady flows using $(r,\ \psi)$ co-ordinates

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1. Abstract

In this paper we have made an approach to find the exact solution of steady, infinite conducting, non-viscous MHD fluid in a rotating frame by the application of Martin's method. As the governing equations are non-linear partial differential equation, a method is developed to convert these equations into solvable form by employing differential geometry where, in the plane of flow, the curvilinear co-ordinate (ϕ, ψ), co-ordinate $\psi = \ \mathit{constant}$ are the streamlines of flow and the co-ordinate lines $\phi =$ constant are left arbitrary. The polar representation of the streamline patterns for these flows are of the form $\frac{\theta - b(r)}{c(r)} =$ constant, are taken and the exact solution for pressure function, velocity vector, vorticity and current are found.

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2. Introduction

Martin [1] introduced a new technique to solve the equations of motion of fluids with infinite electrical conductivity. When the polar representation of the streamline patterns for these flows is of the form $\frac{\theta - b(r)}{c(r)} =$ constant, this method is highly helpful in determining the precise solution of plane viscous, stable, incompressible magnetohydrodynamics (MHD) aligned flows . The future liquid, super conducting liquid metal, is also used in this manner. The exact solution of a steady infinite conducting non-viscous MHD fluid in a rotating frame has been discovered in this study. Understanding the concept of rotating fluid is crucial for understanding some fields, including limnology, oceanography, meteorology, and atmospheric science. . This method involves using a natural curvilinear co-ordinate system in the physical plane (x, y) where $\psi =$ constant are streamlines and $\phi =$ constant is an arbitrary family of curves. By assuming the arbitrary family of curves to be, Chandna and Labropulu [2], analysed the planar viscous flows and discovered the precise solution. Using this technique, Labropulu and Chandna [3] were able to derive the precise solution of stable plane MHD aligned flows by choosing either $\xi(x, y) =$ constant or $\eta(x, y) =$ constant as the arbitrary family of curves$\phi(x, y) =$ constant, where $\xi(x, y) + i\ \eta(x, y)$ is an analytic function of the form $z = x + iy$. Martin's approach was used by M. R. Garg, R.M. Barron, and O.P. Chandna [4] to study plane compressible MHD flows and discover an exact solution. Martin's equation was also employed by K.V. Govinda Raju [5] to discover the answer to the flow of a viscous fluid. Martin's approach was used by S.A. Ali, A. Arora, and N.A. Khan [6] to research fluids in second grade. Martin's approach was used by C. Thakur, M. Kumar, and M.K. Khan [7] to study constantly inclined viscous MHD fluid. Exact solutions for steady plane flows of incompressible fluid with variable viscosity (ε, ψ) or (η, ψ)coordinates were found by C. Thakur, T.P. Singh, M.K. Mahan, et al. [8], R.K. Naeem et al. [9]. M. Kumar, S. Sil [11] , Sayantan Sil, Mantu Prajapati and Manoj Kumar [13] found the exact solution by using Martin's method for rotating fluid.

In this essay, we offer and respond to the two questions listed below.

(i) Can fluid flow along a family of planar curves with $\frac{\theta-b(r)}{c(r)}$ =constants?

(ii) What is the precise integral of the flow defined by the given streamline pattern, given a family of streamlines, $\frac{\theta-b(r)}{c(r)}$ =constant?

Assume fluid flows along the family of provided curves $\frac{\theta-b(r)}{c(r)}$ =constant. Since the streamline function $\psi(r,\ \theta)$ =constant as well along these curves, it follows that there exists some function $R(\psi)$ such that

$$\frac{\theta - b(r)}{c(r)} = R(\psi), \quad R'(\psi) \neq 0 \tag{1}$$

For this work, the curve ϕ =constant are taken to be r =constant onus. Thus, the $(r,\ \psi)$ co-ordinate system is used. Taking $v(r,\ \theta)$ and $u(r,\ \theta)$ to be the components of velocity vector field in polar co-ordinates, we have

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{1}{Rc(r)R'}, \quad v = -\frac{\partial\psi}{\partial r} = \frac{1}{R'}\left[\frac{\theta c'(r)}{c^2(r)} + \left(\frac{b(r)}{c(r)}\right)'\right] \tag{2}$$

3. Flow Equation

In the presence of a magnetic field, the following equations regulate the flow of a viscous, incompressible, spinning, and electrically conducting fluid:

$$div(\vec{V}) = 0 \tag{3}$$

$$\rho\left[(\vec{V}\cdot grad)\vec{V}\right] + 2(\vec{\Omega}\times\vec{V}) + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) + grad\,P = \eta\nabla^2\vec{V} + \mu(curl\,\vec{H})\times\vec{H} \tag{4}$$

$$\frac{1}{\mu\sigma}\,\text{curl}(\text{curl}\ \vec{H}) = \text{curl}(\vec{V}\times\vec{H}) \tag{5}$$

Here $\vec{V} = u\,\hat{i} + v\,\hat{j}$=velocity vector, $\vec{H}$ =magnetic field, P =pressure function, ρ =fluid density, σ =electrical conductivity η =coefficient of viscocity μ =magnetic permeability, Ω =angular velocity.

Here the magnetic field $\vec{H}$ satisfies an additional equation

$$div(\vec{H}) = 0 \tag{6}$$

In the case of aligned flow, the magnetic field is everywhere parallel to the velocity field.

$$\vec{H} = \gamma\vec{V} = \gamma u\,\hat{i} + \gamma v\,\hat{j} \tag{7}$$

Where γ is some unknown function such that

$$\vec{V}\cdot grad\gamma = 0 \tag{8}$$

Let us defined

$$Vorticity function, \omega = v_x - u_y \tag{9}$$

$$Current function, Q = H_{2x} - H_{1y} \tag{10}$$

$$Energy function, E = \frac{1}{2}\rho(u^2+v^2) + P' \tag{11}$$

Where $P' = P - \frac{1}{2}\rho|\vec{\Omega}\times\vec{r}|^2$ =reduced pressure. Where $u(x,\ y)$ *and* $v(x,\ y)$ are the velocity components and $H_1(x,\ y),\ H_2(x,\ y)$ the magnetic field components.
In this work, we deal with infinitely conducting fluids Hence equation (5) now becomes

$$\vec{\nabla}\times(\vec{V}\times\vec{H}) = 0 \tag{12}$$

Separating into components equation (3), (4), (6) and (12) gives and using (7) , (8) and (10) we get

$$u_x + v_y = 0\ (continuity) \tag{13}$$

$$E_x + \eta\omega_y - 2\rho\Omega v - \rho v\Omega - \mu\gamma vQ = 0 \tag{14}$$

$$E_y - \eta\omega_x + 2\rho\Omega v - \rho v\Omega - \mu\gamma uQ = 0 \tag{15}$$

$$u\gamma_x + v\gamma_y = 0\ (solenoidal) \tag{16}$$

$$\omega = v_x - u_y\ (vorticity) \tag{17}$$

$$\gamma\omega + v\gamma_x - u\gamma_y = Q\ (Currentdensity) \tag{18}$$

Above are six partial differential equations in six unknown functions $u(x,\ y),\ v(x,\ y),\ \gamma(x,y), \omega(x,\ y), Q(x,y)$ *and* $E(x,\ y)$. once a solution of this system is determined, the pressure function and the magnetic vector field are found by using (7) and (11)
The equation of continuity (13) implies the existence of a stream function $\psi = \psi(x,\ y)$ such that

$$\psi_x = -v, \quad \psi_y = u \tag{19}$$

We take $\phi(x,\ y)$ =constant to be some arbitrary family of curves which generates with the streamline $\psi(x,\ y)$ = constant a curvilinear net, so that in the physical plane the independent variables x, y can be replace by ϕ and ψ .

$$x = x(\phi,\ \psi),\ \ y = y(\phi,\ \psi) \tag{20}$$

Define a curvilinear net in the (x, y)-plane with the squared element of arc length along any curve is given by

$$ds^2 = L(\phi,\ \psi)\, d\phi^2 + 2M(\phi,\ \psi)\, d\phi d\psi + N(\phi,\ \psi)\, d\psi^2 \tag{21}$$

$$L = {x_\phi}^2 + {y_\phi}^2, \quad M = x_\phi x_\psi + y_\phi y_\psi \quad N = {x_\psi}^2 + {y_\psi}^2 \tag{22}$$

Equation (20) can be solved to obtain $\phi = \phi(x,\ y), \psi = \psi(x,\ y)$, such that

$$x_\phi = J\psi_y,\ \ x_\psi = -J\phi_y, \quad y_\psi = J\phi_x,\ \ y_\phi = -J\psi_x \tag{23}$$

Provided $0 < |J| < \infty$ m where J is the transformation Jacobian and

$$J = x_\phi y_\psi - x_\psi y_\phi = \pm\sqrt{LN - M^2} = \pm W \tag{24}$$

If α be the local angle of inclination of the tangent to the co-ordinate line ψ =constant, directed in the sense of increasing ϕ , we have from differential geometry
$x_\phi = \sqrt{L}\cos\alpha,\ \ y_\phi = \sqrt{L}\sin\alpha,\ \ x_\phi = \frac{M}{\sqrt{L}}\cos\alpha - \frac{J}{\sqrt{L}}\sin\alpha$
$y_\psi = \frac{M}{\sqrt{L}}\sin\alpha + \frac{J}{\sqrt{L}}\cos\alpha,\ \ \alpha_\phi = \frac{J}{L}{\Gamma_{11}}^2,\ \ \alpha_\psi = \frac{J}{L}{\Gamma_{12}}^2$
And $\frac{1}{\omega}\left[\left(\frac{W}{E}{\Gamma_{11}}^2\right)_\psi - \left(\frac{W}{E}{\Gamma_{12}}^2\right)_\phi\right] = 0$

$$Here\ \ {\Gamma_{11}}^2 = \frac{1}{2\omega^2}\left[-MM_\phi + 2LM_\phi - LL_\psi\right] \tag{25}$$

$\Gamma_{12}^2 = \frac{1}{2\omega^2}\left[LN_\phi - ML_\psi\right]$

For infinitely conducting flows, following Martin and Chandna and Labropulu , we transform system of equation (13) – (18) into the -co-ordinates by using (19) and considering (23), as follows.

$$E_\phi x_\psi - E_\psi y_\phi + \eta\left(\omega_\phi x_\psi + \omega_\psi x_\phi\right) - \left[\rho(2\Omega + \omega) - \mu\gamma Q\right] y_\phi = 0 \tag{26}$$

$$E_\phi x_\psi - E_\psi x_\phi + \eta\left(\omega_\phi y_\psi - \omega_\psi y_\phi\right) - \left[\rho(2\Omega + \omega) - \mu\gamma Q\right] x_\phi = 0 \tag{27}$$

Multiplying (26) by x_ϕ and (27) by ψ_ϕ and subtracting, we get

$$JE_\phi = \eta\left[M\omega_\phi - L\omega_\psi\right] \tag{28}$$

Again multiplying equation (26) by x_ϕ and (27) by y_ϕ and subtracting

$$JE_\psi = \eta\left[-M\omega_\psi + N\omega_\phi\right] - J\left[\rho(2\Omega+\omega) - \mu\gamma Q\right] \tag{29}$$

again using (19) in the equation (16) and transforming the resulting equation to $(\phi,\ \psi)$ co-ordinate we get

$$\psi_y\left[\gamma_\phi\phi_x + \gamma_\psi\psi_x\right] - \psi_x\left[\gamma_\phi\phi_y + \phi_\psi\psi_y\right] = 0 \tag{30}$$

Which on simplification gives $E_\phi = 0$ (solenoidal equation)
Now using (19) in equation (18) we get

$$\gamma\omega - \psi_x\gamma_x - \psi_y\gamma_y = Q \tag{31}$$

Using (23) and employing (31) and (22) we get

$$\gamma\omega - \frac{L}{J^2}\gamma_\psi = Q,\ \ (Current density equation) \tag{32}$$

$$And\ \frac{1}{\omega}\left[\left(\frac{W}{E}{\Gamma_{11}}^2\right)_\psi - \left(\frac{W}{E}{\Gamma_{12}}^2\right)_\phi\right] = 0 \tag{33}$$

So these are six equations (28) to (33) for seven functions L, M, N, E, J, ω *and* γ in terms of $(\phi,\ \psi)$.

Here the Christofell symbols ${\Gamma_{11}}^2$ *and* ${\Gamma_{12}}^2$ are given by (25). Martin obtained the necessary and sufficient conditions for the flow of a fluid along the co-ordinate lines ψ =constant of curvilinear co-ordinate system with ds^2 given by (21) to satisfy the principle of conservation mass to be

$$WV = \sqrt{L},\ \ u + iv = \frac{\sqrt{L}}{W}e^{i\alpha} \tag{34}$$

Using the inlegrability conditions $E_{\phi\psi} = E_{\psi\phi}$ in the Linear momentum equations(28) and (29), we find that the unknown function $L(\phi,\ \psi)$, $M(\phi,\ \psi)$, $N(\phi,\ \psi)$, $Q(\phi,\ \psi)$, $\omega(\phi,\ \psi)$ and $\gamma(\psi)$ satisfy the following equations.

$$\frac{1}{\omega} = \frac{1}{W}\left[\left(\frac{M}{W}\right)_\phi - \left(\frac{L}{W}\right)_\psi\right] \tag{35}$$

$$Q = \gamma\omega - \frac{L}{J^2}\gamma_\psi \tag{36}$$

$$\left(\frac{W}{L}{\Gamma_{11}}^2\right)_\psi - \left(\frac{W}{L}{\Gamma_{12}}^2\right)_\psi = 0 \tag{37}$$

$$And,\ \gamma_\phi = 0 \tag{38}$$

$$\eta W\Delta_2\omega + \left(\mu\gamma Q - \rho(2\Omega+\omega\right)_\phi = 0 \tag{39}$$

$$where,\ \ \Delta_2\omega = \frac{1}{W}\left[\left(\frac{N}{W}\omega_\phi - \frac{M}{W}\omega_\psi\right)_\phi + \left(\frac{L}{W}\omega_\phi - \frac{M}{W}\omega_\phi\right)_\psi\right] + \left[\mu\gamma Q - \rho(2\Omega+\omega)_\phi\right] \tag{40}$$

Defines the bettrami's differential operator of 2nd order.
Equations (35) to (39) form an undetermined system Since the co-ordinate lines constant have been left arbitrary. This undetermined system can be made determined in a number of different ways and one such possible ways is to let $\phi(x,\ \psi) = \theta(x,\ y)$ where $(r,\ \theta)$ is the polar co-ordinate system.

4. Exact Solution

To analyse whether a given family of curves $\frac{\theta-b(r)}{c(r)}$ =constant can or cannot be the streamlines, we assume the alternative so that there exists some function (ψ) such that

$$\frac{\theta-b(r)}{c(r)}=R(\psi),\quad R^{'}(\psi)\neq 0 \tag{41}$$

Where $R'(\psi)$ is the derivative of the unknown function (ψ).
Using $\phi=r\ and$ x=rcos$\,\theta$, $y=r\sin\theta$ in (22) we find L, M, N and J in $(r,\ \psi)$.

$$L=1+r^2\left[b^{'}(r)+c^{'}(r)\,R(\psi)\right]^2\qquad N=r^2c^2(r)\,R^{'2}(\psi)$$

$$M=r^2\left[b^{'}(r)+c^{'}(r)\,R(\psi)\right]\,c(r)\,R^{'}(\psi),\quad J=W=rc(r)\,R^{'}(\psi) \tag{42}$$

5. Example-01

Let us now consider the flow with $\theta-a_1r^2-a_2r=$ constant as streamlines. It follows their exists some functions such that

$$\theta=a_1r^2+a_2r+R(\psi),\ with\ a_1\neq 0,\ R(\psi)\neq 0 \tag{43}$$

Comparing (41) with (43), we get
b(r)=a$_1r^2+a_2r,\ \ c(r)=1$,

using these forms of $b(r)$ and $c(r)$ in (42), we get
L=1+r$^2[2a_1r+a_2]^2\quad M=r^2\left[2a_1r+a_2\right]\,R'(\psi)$

$$N=r^2R^{'2}(\psi)\qquad J=W=rR^{'}(\psi) \tag{44}$$

Hence equation (29), (28) (35) and (36) in $(r,\ \psi)$ co-ordinate

$$E_\psi=\eta\left[rR^{'}(\psi)\omega_r-r(2a_1r+a_2)\omega_\psi\right]+\left[\mu\gamma Q-\rho(2\Omega+\omega\right] \tag{45}$$

$$E_r=\eta\left[(2a_1r^2+a_2r)\omega_r-\left(\frac{1}{r}+4a_1^2r^3+a_2^2r+4a_1a_2r^2\right)\frac{1}{R'(\psi)}\omega_\psi\right] \tag{46}$$

$$\omega=\frac{1}{r^2R'^3(\psi)}\left[(4a_1r^2+a_2r)\,R^{'2}(\psi)+R^{''}(\psi)+r^2(2a_1r+a_2)^2R^{''}(\psi)\right] \tag{47}$$

$$Q=\gamma(\psi)\omega-\left[\frac{1+r^2(2a_1r+a_2)^2}{r^2R'^2(\psi)}\right]\gamma^{'}(\psi) \tag{48}$$

Now putting the value of $\omega_\psi,\ \omega_r,\ Q_r,\ \omega_{rr},\ \omega_{r\psi},\ \omega_{\psi\psi},\ \omega_{\psi r}$ in (45) and (46) and making use of the fact $E_{r\psi}=E_{\psi r}$, we get

$$\sum_{n=-3}^{n=5}A_n(\psi)r^n=0 \tag{49}$$

$$Where A_{-3}=\left[4-\frac{2\mu\gamma^2}{\eta}\frac{1}{R'(\psi)}+\frac{2\rho}{\eta}\frac{1}{R'(\psi)}\right]\frac{R^{''}(\psi)}{R'^2(\psi)}+\left[\frac{1}{R'(\psi)}\left(\frac{R^{''}(\psi)}{R'^3(\psi)}\right)\right]^{'} \tag{50}$$

$$A_{-2}=a_2+3a_2\left(\frac{R^{''}(\psi)}{R'^3(\psi)}\right)^{'}+\left(\frac{\rho-\mu\gamma^2}{\eta}\right)\frac{a_2}{R'(\psi)}-a_2\frac{1}{R'(\psi)}\left(\frac{R^{''}(\psi)}{R'^2(\psi)}\right)^{'}+2a_1\left(\frac{R^{''}(\psi)}{R'^2(\psi)}\right)^2+a_2\left(\frac{R^{''}(\psi)}{R'^3(\psi)}\right)^{'} \tag{51}$$

$$A_{-1} = -a_2^2 \frac{R^{''}(\psi)}{R'^2(\psi)} + a_1 \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' - 4a_1 \frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)' + 2a_1 \left[\frac{1}{R'(\psi)} \left(\frac{R^{''}}{R'^3} \right)' \right] \tag{52}$$

$$\mathrm{A}_0 = 8a_1a_2 \frac{R^{''}(\psi)}{R'^2(\psi)} + 8a_1a_2 \left[\frac{1}{R^{''}(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' \right]' + \left[\frac{4a_1a_2\mu\gamma^2}{\eta} - \frac{8a_1^2\rho}{\eta} - \frac{4a_1a_2\rho}{\eta} \right] \frac{R^{''}(\psi)}{R'^3(\psi)}$$
$$- \frac{4a_1a_2\mu\gamma\gamma^{'}}{\eta} \frac{1}{R'^2(\psi)} - a_1^3 \left[\frac{1}{R'(\psi)} \left(\frac{R^{''}}{R'^2} \right)' - \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)^2 \right] \tag{53}$$

$$\mathrm{A}_1 = (28a_1^2 + 4a_1) \frac{R^{''}(\psi)}{R'^2(\psi)} \left[-6a_1a_2^2 + 6a_1a_2 - \frac{4a_1^2}{R'(\psi)} \right] \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' + \frac{8a_1^2\mu\gamma^2}{\eta} \frac{R^{''}(\psi)}{R'^3(\psi)} - \frac{8a_1^2\mu\gamma\gamma'}{\eta} \frac{1}{R'^2(\psi)}$$
$$4a_1^2 \left[-\frac{R^{''}(\psi)}{R'^2(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' - \frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)^{''} \right] - (2a_1a_2^2 + 6a_1a_2) \frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)'$$
$$+(2a_1a_2^2 + 6a_1a_2) \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)^2 - (4a_1^2 + a_2^4) \frac{R^{''}(\psi)}{R'^2(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' \tag{54}$$

$$\mathrm{A}_2 = -24a_1^2a_2 \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' - 20a_1^2a_2 \frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)'$$
$$+8a_1a_2^3 \left[\frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)^{\prime\prime} - \frac{R^{''}(\psi)}{R'^2(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' \right] + 20a_1^2a_2 \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)^2 \tag{55}$$

$$A_3 = 24a_1^2a_2^2 \left[\frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right) \right]' + 24a_1^3 \left(\frac{R^{''}(\psi)}{R'^2(\psi)} \right)^2 \tag{56}$$

$$A_4 = 32a_1^3a_2 \left[\frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' \right]' \tag{57}$$

$$A_5 = 16a_1^4 \left[\frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' \right]' \tag{58}$$

Equation (49) is a polynomial of degree 5 with coefficient as function of r only. Since r and ψ are independent variables, it follows that this can only hold true for all values of r if the coefficients of different power of r vanish simultaneously and we have $A_n = 0$.
Therefore, from (58),

$$\left[\frac{1}{R'(\psi)} \left(\frac{R^{''}(\psi)}{R'^3(\psi)} \right)' \right]' = 0 \tag{59}$$

putting this in (56), we get (for $A_3 = 0$),

$$\frac{R^{''}(\psi)}{R'^2} = 0 \tag{60}$$

Or , $R^{''}(\psi) = 0$, which on integration gives

$$R(\psi) = m_1\psi + \psi_0 \tag{61}$$

Now using (59) and (60), from (55) (for $A_2 = 0$)

$$\left(\frac{R^{''}}{R'^3} \right) = 0 \tag{62}$$

Using (59), (60), (62) in (54), (for $A_1 = 0$)

$$\frac{8a_1^2\mu\gamma^2}{\eta}\frac{R^{''}(\psi)}{R'^3(\psi)} - \frac{8a_1^2\mu\gamma\gamma'}{\eta}\frac{1}{R'^2(\psi)} = 0$$

or,

$$\left(\frac{8a_1\mu\gamma^2}{\eta}\frac{R^{''}(\psi)}{R'^2(\psi)} - \frac{8a_1^2\mu\gamma\gamma'}{\eta}\right)\frac{1}{R'(\psi)} = 0$$

$$\gamma' = 0 \qquad \gamma = \gamma_0 \tag{63}$$

For $A_{-2} = 0$,

$$a_2\left(\frac{\rho - \mu\gamma^2}{\eta}\right) \times \frac{a_2}{R'(\psi)} = 0$$

or,

$$\frac{\rho - \mu\gamma^2}{\eta} \times \frac{1}{R'(\psi)} + 1 = 0 \tag{64}$$

From (61) and (64) and (63)

$$\frac{\rho - \mu\gamma^2}{\eta} \times \frac{1}{m_1} = -1$$

$$m_1 = \frac{\mu\gamma_0^2 - \rho}{\eta}$$

$$\therefore\ R(\psi) = \frac{1}{\eta}(\mu\gamma_0^2 - \rho)\psi + \psi_0 \tag{65}$$

From (47)

$$\omega = \frac{1}{r^2}\left[\left(4a_1r^2 + a_2r\right)\frac{1}{R'(\psi)} + \left[1 + r^2(2a_1r + a_2\right]^2 \times \frac{R^{''}(\psi)}{R'^3(\psi)}\right]$$

$$\omega = \left(4a_1 + \frac{a_2}{r}\right)\frac{\eta}{\mu\gamma_0^2 - \rho} \tag{66}$$

From (48)

$$Q = \gamma(\psi)\omega - \left[\frac{1 + r^2(2a_1r + a_2)^2}{r^2R'^2(\psi)}\right]\gamma'(\psi)$$

$$Q = \gamma_0\omega \quad or \quad Q = \frac{\eta\gamma_0\left(4a_1 + \frac{a_2}{r}\right)}{\mu\gamma_0^2 - \rho} \tag{67}$$

Also by using $dE = \frac{\partial E}{\partial r}dr + \frac{\partial E}{\partial \psi}d\psi$, and on integrating both site, we get

$$E = \left(\frac{\eta^2 a_2}{\rho - \mu\gamma_0^2}\right)\left[6a_1a_2r + {a_2}^2\ln r - 2\rho\Omega a_2r - 4a_1(\theta - -a_1r^2\psi_0)\right] \tag{68}$$

Also

$$u = \frac{1}{rc(r)R''(\psi)} = \frac{1}{rm_1} = \frac{\eta}{\mu\gamma_0^2 - \rho} \times \frac{1}{r} \tag{69}$$

And

$$v = \frac{1}{R'(\psi)}\left[\frac{\theta c'(r)}{c^2(r)} + \left(\frac{b(r)}{c(r)}\right)'\right]$$

$$v = \frac{\eta}{\mu\gamma_0^2 - \rho}(2a_1r + a_2) \tag{70}$$

$$\therefore\ u^2+v^2=\frac{\eta^2}{\left(\mu\gamma_0^2-\rho\right)^2}\left[\frac{1}{r^2}+4a_1^2r^2+a_2^2+4a_1a_2r\right] \tag{71}$$

Hence ,

$$P=E-\frac{1}{2}\rho V^2$$

0n solving, we get

$$P=\frac{\eta^2}{\mu{\gamma_0}^2-\rho}\left[6a_1a_2r+{a_2}^2\log r-2\rho\Omega a_2r-4a_1\left(\theta-a_1r^2-\psi_0\right)\right]$$

$$+\frac{\rho\eta^2}{2\left(\mu{\gamma_0}^2-\rho\right)}\left[\frac{1}{r^2}+4{a_1}^2r^2+{a_2}^2\right]+p_0 \tag{72}$$

6. Example-02

Let us now consider a different flow with $\theta-ar=$ constant as streamlines, it follows their exist some function $R(\psi)$ such that$\theta=ar+R(\psi)$, with

$$a\neq0, R'(\psi)\neq0 \tag{73}$$

Comparing equation (41) with (73) , we get

$$b(r)=ar,\ \ c(r)=1$$

, using these forms of $b(r)$ and $c(r)$ in equation (42) we get

$$L=1+a^2r^2,\quad M=ar^2R'(\psi)\ ,\quad N=r^2{R'}^2(\psi)\,,\quad J=W=rR'(\psi) \tag{74}$$

Hence equation (28), (29),(35) and (36) in (r,ψ) co-ordinate

$$E_r=\eta\left[ar\omega_r-\left(\frac{1}{r}+a^2r\right)\frac{1}{R'(\psi)}\omega_\psi\right] \tag{75}$$

$$E_\psi=\eta\left[-ar\omega_\psi+rR'(\psi)\,\omega_r\right]+\left[\mu\gamma Q-\rho\left(2\Omega+\omega\right)\right] \tag{76}$$

$$\omega=\frac{a}{rR'(\psi)}+\frac{1+a^2r^2}{r^2}\times\frac{R''(\psi)}{R'''^3} \tag{77}$$

$$Q=\gamma\omega-\frac{1+a^2r^2}{r^2R'^2}\gamma' \tag{78}$$

Now putting the value of $\omega_\psi,\ \omega_r,\ Q_r,\ \omega_{rr},\ \omega_{r\psi},\ \omega_{\psi\psi},\ \omega_{\psi r}$, in (45) and (46) and making use of the fact $E_{r\psi}=E_{\psi r}$, we get

$$\sum_{n=-3}^{n=1}A_n(\psi)r^n=0 \tag{79}$$

where,

$$A_{-3}=\left[\begin{array}{l}\left(\frac{1}{R'}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right)'+2R'(\psi)\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'-6R'(\psi)\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\\+\frac{2\mu\gamma^2}{\eta}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'-\frac{2\mu\gamma}{\eta}\frac{\gamma'}{R'^2(\psi)}-\frac{2\rho}{\eta}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\end{array}\right]$$

$$A_{-2}=\left[\begin{array}{l}-2a\left(\frac{R''(\psi)}{R'^3(\psi)}\right)''+a\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)'-a\left(\frac{R''(\psi)}{R'^2(\psi)}\right)^2+a-2a\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\\+a\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'-2a+\frac{\mu\gamma^2 a}{\eta}\frac{1}{R'(\psi)}-\frac{a\rho}{\eta}\frac{1}{R'(\psi)}\end{array}\right]$$

$$A_{-1}=\left[a^2\frac{R''(\psi)}{R'^2(\psi)}+2a^2\left(\frac{1}{R'}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right)'\right]$$

$$A_0=a^3\frac{1}{R'}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)'-a^3\left(\frac{R''(\psi)}{R'^2(\psi)}\right)^2+a^3\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'$$

$$A_1=a^4\left(\frac{1}{R'}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right)'$$

Now for A_1 to be zero,

$$\left(\frac{1}{R'}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right)'=0 \tag{80}$$

For A_{-1} to be zero,

$$\frac{R''(\psi)}{R'^2(\psi)}=0 \text{ or, } R''(\psi)=0 \text{ or } R(\psi)=m_1\psi+\psi_0 \tag{81}$$

For A_0. to be zero,

$$\frac{R''(\psi)}{R'^3(\psi)}=0 \tag{82}$$

For A_{-2} to be zero,

$$-a+a\left(\frac{\mu\gamma^2-\rho}{\eta}\right)\frac{1}{R'(\psi)}=0\Rightarrow\left(\frac{\mu\gamma^2-\rho}{\eta}\right)\frac{1}{m_1}=1$$

Hence,

$$m_1=\left(\frac{\mu\gamma^2-\rho}{\eta}\right) \tag{83}$$

And for A_{-3} to be zero,

$$\frac{\gamma'}{R'^2(\psi)}=0\Rightarrow\gamma'=0\Rightarrow\gamma=\gamma_0 \tag{84}$$

Now from (77),

$$\omega=\frac{a\eta}{(\mu\gamma^2-\rho)\,r} \tag{85}$$

And from (78),

$$Q=\gamma\omega=\gamma_0\frac{a\eta}{(\mu\gamma^2-\rho)\,r} \tag{86}$$

Also,

$$u=\frac{\eta}{(\mu\gamma^2-\rho)\,r},\quad v=\frac{\eta a}{(\mu\gamma^2-\rho)}. \tag{87}$$

And,

$$\text{P}=\frac{a^2\eta^2}{(\mu\gamma^2-\rho)}\log r-\frac{\rho\eta^2}{2(\mu\gamma^2-\rho)^2r^2}-2\rho a_2\Omega r+p_0 \tag{88}$$

7. Example 03

Let us now consider a different flow with $\theta - a_1r^3 - a_2r^2 =$ constant as streamlines. It follows that their exists some function $R(\psi)$, such that

$$\theta = a_1r^3 - a_2r^2 + R(\psi)\,, \text{ with } a_1, a_2 \neq 0,\; R^{'}(\psi) = 0 \tag{89}$$

Hence, b$(r) = a_1r^3 + a_2r^2,\; c(r) = 1, therefor$
$L = 1 + r^2\left[3a_1r^2 + 2a_2r\right]^2, \quad N = r^2R^{'2}(\psi)$
$M = r^2\left[3a_1r^2 + 2a_2r\right]\gamma^{'}(\psi)\,, \quad J = R^{'}(\psi)$
Hence equation (28),(29),(35) and (36) in (r,ψ)co-ordinate

$$E_r = \eta\left[\left(3a_1r^3 + 2a_2r^2\right)\omega_r - \left(\frac{1}{r} + 9{a_1}^2r^5 + 4{a_2}^2r^3 + 12a_1a_2r^4\right)\frac{1}{R'(\psi)}\omega_r\right] \tag{90}$$

$$E_\psi = \eta\left[\left(-3a_1r^3 - 2a_2r^2\right)\omega_\psi + rR'(\psi)\,\omega_r\right] + \left[\mu\gamma Q - \rho\left(2\Omega + \omega\right)\right] \tag{91}$$

$$\omega = \left[\frac{1}{R'(\psi)}\left(9a_1r + 4a_2\right) + \frac{1}{r^2}\frac{R''(\psi)}{R'''^3(\psi)} - \left(9{a_1}^2r^4 + 4{a_2}^2r^2 + 12a_1a_2r^3\right)\frac{R''(\psi)}{R'''^3(\psi)}\right] \tag{92}$$

$$Q = \gamma\omega - \left[\frac{1}{r^2R'^2(\psi)} + \frac{1}{R'^2(\psi)}\left(3a_1r^2 + 2a_2r\right)^2\right]\gamma' \tag{93}$$

Now putting the value of $\omega_\psi,\ \omega_r,\ Q_r,\ \omega_{rr},\ \omega_{r\psi},\ \omega_{\psi\psi},\ \omega_{\psi r}$ in (45) and (46) and making use of the fact $E_{r\psi} = E_{\psi r}$, we get

$$\sum_{n=-3}^{n=9} A_n(\psi)r^n = 0 \tag{94}$$

Where,

$$A_{-3} = 4\frac{R''(\psi)}{R'^2(\psi)} - \frac{2\mu\gamma^2}{\eta}\frac{R''(\psi)}{R'^3(\psi)} + \frac{2\rho}{\eta}\frac{R''(\psi)}{R'^3(\psi)} + \left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

$$A_{-1} = 4a_2\frac{R''(\psi)}{R'^3(\psi)} - 4a_2\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)' + 4a_2\left(\frac{R''(\psi)}{R'^2(\psi)}\right)'$$

$$A_0 = -9a_1\left(\frac{R''(\psi)}{R'^3(\psi)}\right)' + 6a_1\left(\frac{R''(\psi)}{R'^3(\psi)}\right)' + 9a_1 + \frac{9a_1\mu\gamma^2}{\eta R'(\psi)} - \frac{9\rho a_1}{\eta R'(\psi)} - \frac{9a_1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'$$

$$A_1 = \frac{-8\mu\gamma^2{a_2}^2}{\eta}\frac{R''(\psi)}{R'^3(\psi)} + \frac{8\rho{a_2}^2}{\eta}\frac{R''(\psi)}{R'^3(\psi)} - 4{a_2}^2\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]' + 8{a_2}^2\mu\gamma\frac{\gamma'}{R'^2(\psi)} + 4a_2\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

$$A_2 = -36\frac{\mu\gamma^2}{\eta}a_1a_2\frac{R''(\psi)}{R'^3(\psi)} + \frac{36\rho a_1a_2}{\eta}\frac{R''(\psi)}{R'^3(\psi)} + 24a_1a_2\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

$$A_3 = -36{a_1}^2\frac{R''(\psi)}{R'^2(\psi)} + 32{a_2}^3\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\left(27{a_1}^2 - 36\frac{\mu\gamma^2{a_1}^2}{\eta} + 36\frac{\rho{a_1}^2}{\eta}\right)\frac{R''(\psi)}{R'^3(\psi)}$$
$$- 16{a_2}^3\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)' + 16{a_2}^3\left(\frac{R''(\psi)}{R'^3(\psi)}\right)^2 + 36\mu\gamma{a_1}^2\frac{\gamma'}{R''^2(\psi)}$$

$$A_4 = -36a_1{a_2}^2\left(\frac{R''(\psi)}{R'^3(\psi)}\right)' + 36a_1{a_2}^2\left(\frac{R''(\psi)}{R'^2(\psi)}\right)' + 48a_1{a_2}^2\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)'$$
$$-84a_1{a_2}^2\left(\frac{R''(\psi)}{R'^2(\psi)}\right)^2 - 72a_1{a_2}^2\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'$$

$$A_5 = 216\left(\frac{R''(\psi)}{R'^3(\psi)}\right)' + 144{a_1}^2a_2\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)' - 144{a_1}^2a_2\left(\frac{R''(\psi)}{R'^2(\psi)}\right)^2$$
$$+16{a_2}^4\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]' - 72{a_1}^2a_2\left(\frac{R''(\psi)}{R'^3(\psi)}\right)$$

$$A_6 = -297{a_2}^3\left(\frac{R''(\psi)}{R'^3(\psi)}\right)' + 81{a_1}^3\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)' - 81{a_1}^3\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^2(\psi)}\right)^2 + 96a_1{a_2}^3\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

$$A_7 = 216{a_1}^2{a_2}^2\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

$$A_8 = 216\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

$$A_9 = 81{a_1}^2\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]'$$

Now for A_9 to be zero,

$$\left[\frac{1}{R'(\psi)}\left(\frac{R''(\psi)}{R'^3(\psi)}\right)'\right]' = 0 \tag{95}$$

For A_2 to be zero,

$$\frac{R''(\psi)}{R'^3(\psi)} = 0 \tag{96}$$

For A_{-3} to zero,

$$\frac{R''(\psi)}{R'^2(\psi)} = 0 \text{ or, } R''(\psi) = 0 \text{ or } R(\psi) = m_1\psi + \psi_0 \tag{97}$$

For A_0 to be zero,

$$9a_1 + \frac{9a_1\mu\gamma^2}{\eta R'(\psi)} - \frac{9\rho a_1}{\eta R'(\psi)} = 0$$

, which gives,

$$m_1 = \frac{\rho - \mu\gamma^2}{\eta} \tag{98}$$

forA_1 to be zero,

$$\frac{\gamma'}{R'^2(\psi)} = 0 \Rightarrow \gamma' = 0 \Rightarrow \gamma = \gamma_0 \tag{99}$$

Now from (92),

$$\omega = (9a_1 + 4a_2)\frac{(\rho - \mu\gamma^2)}{\eta} \tag{100}$$

And from (93),

$$\omega = (9a_1 + 4a_2)\,\frac{\gamma_0\left(\rho - \mu\gamma^2\right)}{\eta} \tag{101}$$

Also,

$$u = \frac{\eta}{\left(\rho - \mu{\gamma_0}^2\right) r},\ v = \left(3a_1 r^2 + 2a_2 r\right)\frac{\eta}{\left(\mu\gamma^2 - \rho\right)} \tag{102}$$

And,

$$\text{P}=\frac{\eta^2}{\left(\rho - \mu\gamma^2\right)^2}\left[\frac{9{a_1}^2}{4}\left(\rho - 3\mu\gamma^2\right) r^4 - \frac{\rho}{2r^2} + 2{a_2}^2\left(\rho - 2\mu\gamma^2\right)\right] - 2\rho a_2 \Omega r + p_0 \tag{103}$$

8. Conclusion

This study intends to introduce the reader to Martine's approach, which uses a coordinate system (r, ψ) to discover the precise solution of an MHD rotating fluid flow. We employ three different stream function types, and in each case we were able to precisely solve the pressure function, vorticity function, etc. These findings are identical to those made by F. Labropulu and O.P. Chandna [2] after applying the $\Omega = 0$. Additionally, if the Magnetograph plane is taken into account for the flow and the co-ordinate(ϕ, ψ) is used in its place (r, ψ), the outcome of M. Kumar and S. Sil [11] can be reached after an appropriate transform.

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