Exact solutions to the two-dimensional non-linear Sine-Gordon equation

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The Sine-Gordon equation is a significant non-linear dispersive equation with applications across diverse areas of science and engineering, including solid-state physics, field theory, and optical systems. In this work, we focus on the two-dimensional Sine-Gordon equation and present three new classes of exact solutions by employing analytical techniques. A particular solution has also been found which is identical to soliton (or kink) solutions obtained by previous authors. These findings provide new insights into the behavior of non-linear wave propagation and can have implications for theoretical studies and practical applications where Sine-Gordon models are relevant.

I. INTRODUCTION

Non-linear differential equations play a vital role in various fields of science and engineering, such as physics, electricity, chemistry, biology, control theory, signal and image processing. These equations are essential for modeling complex systems where linear approximations fail to capture the true nature of dynamic interactions. Among these, the Sine-Gordon (SG) equation stands out as a non-linear hyperbolic partial differential equation with significant applications across numerous scientific fields, such as differential geometry [1, 2], dislocation of metals [3], non-linear optics [4], plasma physics [5], and the propagation of magnetic flux on a Josephson line [6, 7]. Additionally, it plays an important role in developing a unitary theory for elementary particles [8, 9] and quantum field theory [10].

Due to its wide range of applications, the SG equation has garnered continuous attention from researchers, particularly for its rich variety of soliton solutions. A soliton [11, 12] is typically described as a solitary traveling wave that exhibits particle-like behavior, characterized by stability, localizability, and finite energy. Solitons are unique in that when two or more of them collide, they do not break up or disperse. Instead, they emerge from the interaction retaining their original shape and velocity, potentially undergoing only a phase shift. This remarkable property makes solitons valuable in studying non-linear wave phenomena, as they demonstrate how stable structures can persist in non-linear systems, despite complex interactions.

The soliton solutions of the Sine-Gordon equation are stable localized waves that maintain their shape during both propagation and interaction. This stability is one of the equation's most notable features and contributes to its broad applicability in fields such as condensed matter physics, nonlinear optics, and biological systems. The soliton (or kink) solutions, along with anti-soliton (or antikink) and breather solutions of the one-dimensional SG equation, have been thoroughly studied by Debnath in his book [13], where methods such as Bäcklund transformations [14], similarity solutions [15], and the inverse scattering method [14] were employed. These soliton solutions, including the kink and anti-kink, are crucial in comprehending the nonlinear dynamics of various physical systems, as they signify transitions between distinct vacuum in field theory.

The study of the SG equation thus offers deep insights into the behavior of non-linear systems, making it an essential tool for scientists and engineers seeking to model and analyze complex physical phenomena. The standard form of (2+1)-dimensional SG equation is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 1/c^2 \frac{\partial^2 u}{\partial t^2} = Sin u \tag{1}$$

where $u = u(x, y, t)$ and c, the velocity of propagation.

The present study is an investigation towards exact solutions to the equation (1). Several different methods for obtaining exact solutions of nonlinear partial differential equations arising in physical problems have been developed recently. Among them there are sine-cosine method [16], the homogeneous balance method [17–19], the hyperbolic tangent expansion [20–22], the Jacobi elliptic function expansion method [23, 24], the trial function method [25, 26], the nonlinear transformation method [27, 28] and others [29–31]. Recently, the generalized nonlinear Sine-Gordon equation was solved using physics-informed neural networks [32], a machine learning-based method. It incorporate the physical laws governing the equation (through partial differential equations) into the neural network training process, allowing them to efficiently approximate solutions to complex nonlinear systems.

The structure of our paper is as follows. In Section I, we introduce the SG equation mentioning the earlier works. In section II, we present the exact solution of the equation. Further in Sec. III, we discuss the particular solutions. Following this in the Sec. IV we discuss the solutions obtained. Finally, in Sec. V we conclude our work and highlighted diverse application of SG equation in different fields.

II. EXACT SOLUTIONS

We define two new variable ϑ and z, such that

$$\vartheta = ln \mid tan\frac{u}{4} \mid \tag{2}$$

and

$$z = ct \tag{3}$$

Inserting (2) and (3) into equation (1) yields

$$[1 + exp(2\vartheta)](\vartheta_{xx} + \vartheta_{yy} - \vartheta_{zz}) + [1 - exp(2\vartheta)](\vartheta_x^2 + \vartheta_y^2 + \vartheta_z^2 - 1) = 0 \tag{4}$$

where $\vartheta_x = \frac{\partial\vartheta}{\partial x}$ and $\vartheta_{xx} = \frac{\partial^2\vartheta}{\partial x^2}$. One can split the equation (4) to get the following coupled differential equations,

$$\vartheta_x^2 + \vartheta_y^2 - \vartheta_z^2 = 1 \tag{5}$$

$$\vartheta_{xx} + \vartheta_{yy} - \vartheta_{zz} = 0 \tag{6}$$

In [33], Ghosh el. al. offered three exact solutions of the coupled equations of the form (5) and (6), which are appearing in several physical situations. For example, equations of this type occur in two cases, one in the special case of nonlinear field equations for the chiral invariant model of pion dynamics studied by Ray [34] and another in the special situation of the problem of stability of scalar soliton studied by Schiff [35].
Following the works of Ghosh et. al. the equations (5) and (6) admits three classes of solutions, which are:

$$\text{(i)} \quad \vartheta = A\,x + B\,y - S\sqrt{A^2 + B^2 - 1}\,z + D \tag{7}$$

where A, B, D are arbitrary constants and $S = 1$ or -1.

$$\text{(ii)} \quad \vartheta = \alpha\,x + \beta\,y - S\sqrt{\alpha^2 + \beta^2 - 1}\,z + f(\alpha, \beta) \tag{8}$$

where $f(\alpha, \beta)$ is an arbitrary function and $S = 1$ or -1; and α, β which are functions of x, y and z given by

$$-x + \frac{S\alpha z}{\sqrt{\alpha^2 + \beta^2 - 1}} = \frac{\partial f}{\partial \alpha} \tag{8a}$$

$$-y + \frac{S\beta z}{\sqrt{\alpha^2 + \beta^2 - 1}} = \frac{\partial f}{\partial \beta} \tag{8b}$$

and

$$(\alpha_x + \beta_y)\sqrt{\alpha^2 + \beta^2 - 1} + (\alpha\alpha_z + \beta\beta_z) = 0 \qquad (8c)$$

where $\alpha_x = \frac{\partial\alpha}{\partial x}$, $\beta_y = \frac{\partial\beta}{\partial y}$ and so on.

$$\text{(iii)}\quad \vartheta = H(q)x + qy - Sz\sqrt{H^2 + q^2 - 1} + G(q) \qquad (9)$$

where $H(q)$, $G(q)$ are functions of q and q is a function of x, y, z given by

$$x - \frac{H(q)Sz}{\sqrt{H^2 + q^2 - 1}}\left(\frac{q_y}{q_z}\frac{\sqrt{H^2 + q^2 - 1} + Sqq_z}{\sqrt{H^2 + q^2 - 1} + SHq_z}\right) + \frac{Szq}{\sqrt{H^2 + q^2 - 1}} - y = \frac{dG}{dq} \qquad (9a)$$

$$-\frac{q_y}{q_x}\left(\frac{\sqrt{H^2 + q^2 - 1} + Sqq_z}{\sqrt{H^2 + q^2 - 1} + SHq_z}\right) = \frac{dH}{dq} \qquad (9b)$$

Using (2) and (3) from (7), (8) and (9) three set of new exact solution of two dimensional (2D) SG equation (1) are obtained as follows:
The first set of solutions of equation (1) is given by

$$u = 4arctan\left[exp\left(A\,x + B\,y - Sc\sqrt{A^2 + B^2 - 1}\,t + D\right)\right] \qquad (10)$$

The second set of solutions of (1) is given by

$$u = 4arctan\left[exp\left(\alpha\,x + \beta\,y - Sc\sqrt{\alpha^2 + \beta^2 - 1}\,t + f(\alpha, \beta)\right)\right] \qquad (11)$$

subject to the conditions (8a), (8b) and (8c).
The third set of solutions of (1) is given by

$$u = 4arctan\left[exp\left(H(q)\,x + q\,y - Sc\sqrt{H^2 + q^2 - 1}\,t + G(q)\right)\right] \qquad (12)$$

subject to the conditions (9a) and (9b).

III. PARTICULAR SOLUTIONS

In this section the application of our analytical method to non-linear 2D-SG equations of the form (1) is illustrated to find particular solutions. The form of one particular solution will be similar to soliton solution.

Example 1: Consider the non-linear 2D-SG equation (1) subject to the initial conditions.

$$u(x, y, 0) = 4arctan[exp(x + y)] \qquad (13a)$$

$$\text{and } u_t(x, y, 0) = \frac{4}{1 + exp[2(x + y)]} \qquad (13b)$$

where $\Omega = \{(x, y) : a_1 \le x \le b_1, c_1 \le y \le d_1\}$ and $t > 0$.
Using the exact solution of (1) given by (10) and the initial condition (13a), one gets $A = 1$, $B = 1$ and $D = 0$.
Using the Eq. (10) and (13b), we obtain $C = \pm 1$ according as $S = \mp 1$. Thus, a particular solution of (1) is given by

$$u(x, y, t) = 4arctan\,[exp(x + y \pm t)] \qquad (14)$$

The form of the solution given by (14) is identical to the soliton (or kink) solution of 2D-SG equation obtained by Su [36] using localized method of approximate particular solutions (LMAPS). In absence of y in RHS of (14) the form of solution given in Eq. (14) represents the soliton solution of one-dimensional Sine-Gordon equation formed by Debnath [13]. In Fig. 1 surface plot of $u(x, y, t)$ as obtained in Eqn. 14 is shown at different time parameter.

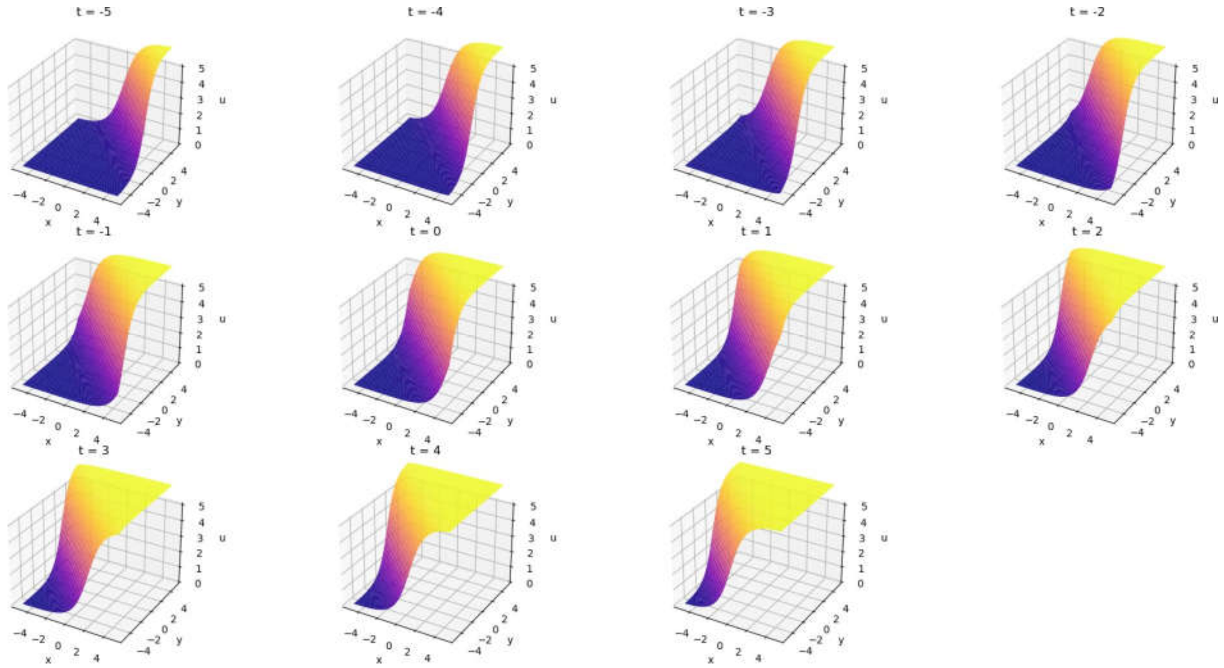

FIG. 1. Time evolution of $u(x, y, t) = 4arctan\,[exp(x + y \pm t)]$: solution obtained as in equation 14.

Example 2: Consider 2D Sine-Gordon equation of the form (1) subject to the initial conditions

$$u(x, y, 0) = 4arctan\left[exp\left(3 - \sqrt{\frac{(x-y)^2}{3} + \frac{(x+y)^2}{2}}\right)\right] \tag{15a}$$

$$\text{and}\quad u_t(x, y, 0) = 4 \tag{15b}$$

where $\Omega = \{(x, y) : a_1 \le x \le b_1, c_1 \le y \le d_1\}$ and $t \ge 0$.

Using the exact solution of (1) given by (10) and the initial condition (15a), we have $A = B = \pm\frac{\sqrt{30}}{6}$ and $D = 3$. From the equation (10) and (15b), one gets $C = \frac{\sqrt{6}}{2}$.

So a particular solution of the equation (1) is

$$u(x, y, t) = 4arctan\left[exp\left(\pm\frac{\sqrt{30}}{6}(x + y) \pm t + 3\right)\right] \tag{16}$$

In Fig. 2 surface plot of $u(x, y, t)$ as obtained in Eqn. 16 is shown at different time parameter.

IV. SUMMARY & DISCUSSIONS

In this paper, we have presented three classes of new different exact solutions of equation (1) given by (10), (11), (12) from which one can study the dynamical behavior of these solutions. In a recent letter, Zagrodzinski [37] found particular solutions of the equation (1) and showed that the general solution of (1) is of the form

$$U = 4arctan[exp(y)\,f(x - t)]$$

But the explicit form of $f(x-t)$ is not given; the first set of solution given by (10) becomes $U = 4arctan\,[exp(y)e^{x-t}]$. Substituting $A = B = C' = 1$ and $D = 0$. where $C' = SC\sqrt{A^2 + B^2}$. So, in this case $f(x - t) = e^{x-t}$. Otherwise,

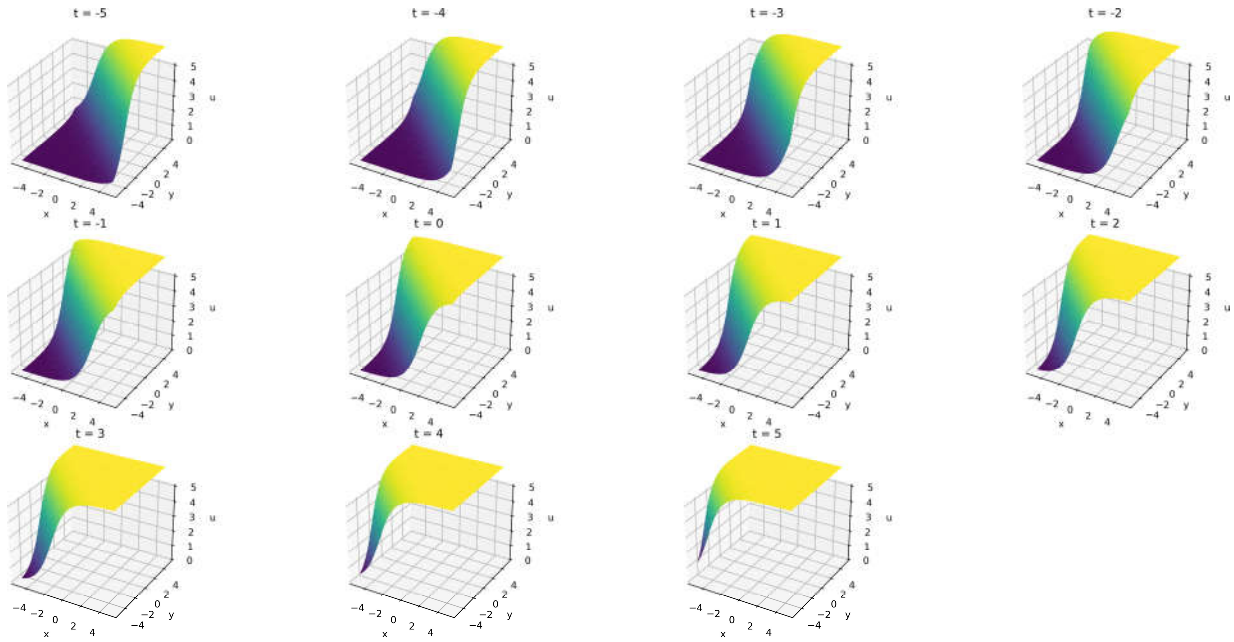

FIG. 2. Time evolution of $u(x, y, t) = 4arctan\left[exp\left(\pm\frac{\sqrt{30}}{6}(x+y) \pm t + 3\right)\right]$: solution obtained as in equation 16.

first set of solutions given by (10) can be written as

$$U = 4arctan\left[exp(By)exp\left(A\,x - SC\sqrt{A^2+B^2}\,t + D\right)\right]$$

Second and third sets of solutions given by (11) and (12) is of the form

$$U = 4arctan\,[exp(C_0 y) \times \text{an exponential function of x and t}].$$

where $C_0 = \beta$ or q.
Thus, we see that the general solution of the equation (1) is $U = 4arctan[exp(k\,y)F(x,t)]$, where $F(x,t)$ is an exponential function of x and t instead of $F(x,t) = f(x-t)$ and k is a constant.

In another letter, Gibbon *et. al.* [38] found a resonant solution of 2D-SG equation or single Sine-Gordon equation which is given by

$$U = 4arctan\left[\sum_{i=1}^{N} exp(\vartheta_i)\right]$$

where $\vartheta_i = p_i x + q_i y - \omega_i t$, $p^2_i + q^2_i - \omega^2_i = m^2$; p_i, q_i, ω_i and m^2 being constant.

Thus, the solution of 2D-SG equation (1) obtained by Gibbon *et. el.* and the first set of solutions given by (10) are equivalent. But the solution given by Eq. (10) is more compact and simpler. Two particular solutions of the equation (1) subject to initial conditions have been obtained. The form of one particular solution is identical to soliton (or kink) solution obtained by previous authors.

V. CONCLUSIONS

The Sine-Gordon equation continues to serve as a cornerstone in the study of wave-like phenomena and complex nonlinear dynamics across a broad spectrum of scientific disciplines. Its mathematical structure and solitonic solutions make it a powerful tool in modeling and understanding a wide range of physical systems [39]. In condensed matter physics, it effectively describes solitons and topological defects [40], which are crucial in determining the behavior of low-dimensional materials. In the realm of nonlinear optics [41], the equation models the propagation of optical pulses through nonlinear media—particularly within optical fibers—where preserving pulse shape is vital for maintaining signal integrity. Similarly, in superconductivity, the Sine-Gordon equation governs the dynamics of Josephson junctions [42, 43], which are integral to the operation of superconducting quantum devices. Furthermore, in biophysics, it is utilized to model biological phenomena [44] such as the transmission of nerve impulses and the conformational changes in proteins.

These diverse applications underscore the remarkable versatility of the SG equation and its capacity to capture essential physical behaviors governed by nonlinear and dispersive effects. Motivated by its significance, this work presents three new, distinct classes of exact solutions to the 2D-SG equation. By applying suitable transformations, the 2D-SG equation is reduced to a system of coupled equations, enabling the construction of explicit solutions in terms of logarithmic and exponential functions. These newly derived solutions enrich the existing body of knowledge by offering generalized and compact representations that encapsulate both known and novel dynamical behaviors. Notably, they include and extend particular solutions previously reported by Zagrodzinski [37] and Gibbon [38], illustrating the coherence of our findings with established literature while also highlighting their broader scope.

The derived solutions provide more than just analytical insight—they serve as foundational tools for studying nonlinear field equations and validating numerical methods. Especially relevant to systems involving soliton dynamics and stability, these solutions open avenues for extending the analysis to more general forms of the Sine-Gordon equation, including those with damping, external forces, or higher dimensions. With the rise of data-driven techniques, such as physics-informed neural networks, these findings also bridge theoretical and computational approaches, offering a path toward broader interdisciplinary applications in nonlinear wave physics.

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DATA AVAILABILITY

No data was used for the research described in the article.

DECLARATION FOR CONFLICT OF INTEREST

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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